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A FINITE ELEMENT EIGENVALUE METHOD FOR SOLVING TRANSIENT HEAT CONDUCTION PROBLEMS

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ABSTRACT

An eigenvalue method is presented for solving the transient heat conduction problem with time-dependent or time-independent boundary conditions. The spatial domain is divided into finite elements and at each finite element node, a closed-form expression for the temperature as a function of time can be obtained. Three test problems which have exact solutions were solved in order to examine the merits of the eigenvalue method. It was found that this method yields accurate results even with a coarse mesh. It provides exact solution in the time domain and therefore has none of the time-step restrictions of the conventional numerical techniques. The temperature field at any given time can be obtained directly from the initial condition and no time-marching is necessary. For problems where the steady-state solution is known, only a few dominant eigenvalues and their corresponding eigenvectors need to be computed. These features lead to great savings in computation time, especially for problems with long time duration. Furthermore, the availability of the closed form expressions for the temperature field makes the present method very attractive for coupled problems such as solid–fluid and thermal–structure interactions.

KEY WORDS Transient heat conduction Eigenvalue method Finite element method

NOMENCLATURE

A, A_w	defined by (10) and (13)
$[A], a_{ij}, a_{lm}^{(e)}$	thermal stiffness matrix or tensor
$a_i, a_l^{(e)}$	vector of nodal heat load resulting from specified nodal temperature
$[B], b_{ij}, b_{lm}^{(e)}$	capacitance matrix or tensor
$b_i, b_l^{(e)}$	vector of nodal heat load resulting from specified time derivative of nodal temperature
$[C], c_{ij}, c_{lm}^{(es)}$	boundary convection/radiation matrix or tensor
$c_i, c_l^{(es)}$	vector of boundary nodal heat load resulting from specified boundary nodal temperature
$c^{(s)}$	normalized constants, defined by (29)
c	specific heat
d_s	defined by (38), dependent on a_i and c_i
E	defined by (8)
E_w	defined by (12) and (17)
e_s	defined by (39), dependent on b_i

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F	local heat flux defined by (4)
$F_{r\infty}$	ambient radiation flux defined by (6)
f	element interpolation function
$\bar{f}$	boundary interpolation function
h	convective heat transfer coefficient, defined by (5)
h_r	radiative heat transfer coefficient, defined by (6)
k	thermal conductivity
M	total number of finite elements
M_2, M_3, M_4	number of boundary elements of the second, third and radiation kinds, respectively
n	outward normal to Ω
p_s	principal coordinates, defined by (30)
$Q_i, Q_i^{(es)}$	defined by (18), (23)
q_i	generalized coordinates, defined by (14) and (19)
q_i	dq_i/dt
R	defined by (9) and (16)
T	temperature
t	time
W	internal heat generation rate per unit volume
x_α	spatial coordinates
$[Y]$, y_{is}	eigenvector matrix defined by (30)

Greek symbols

α_s	defined by (36)
β_s	defined by (37)
Δ	triangular area
θ	$T-T_0$
$\bar{\theta}$	trial surface temperature for iteration in linearized computation of radiative boundary condition
λ_s	eigenvalues
ρ	density
$c^{(s)}\phi_i^{(s)}$	normalized eigenvectors
π_s	defined by (32), dependent on Q_i
Γ	boundary
Ω	domain

Subscripts

g, h	denote condition at known node
i	generalized coordinate index
l, m	denote condition at unknown node
s	eigenmode index
0	initial condition
∞	ambient condition

Superscripts

(e)	finite element index
(es)	boundary element index
(s)	eigenmode index

INTRODUCTION

In conventional finite difference and finite element methods for solving transient heat conduction problems, the time derivative is usually replaced by finite differences and the time-dependent temperature field is obtained by using a step-by-step time-marching scheme. It is well known that the time step in the explicit schemes is severely restricted due to stability considerations, and the implicit schemes involve solving matrix systems and may lead to oscillatory results if the time steps are large. When the temperature field with long time duration is needed, the amount of computation required is considerable. For both explicit and implicit schemes, the time and spatial increments must be chosen with care and these increments are usually quite small to avoid stability and oscillations problems.

A number of numerical techniques have been used to reduce the transient heat conduction equation to a system of ordinary differential equations by discretization in the spatial variables[1]. The discretizations can be finite element approximations (Galerkin and collocation) or finite difference approximations (method of lines). These techniques are then used in conjunction with a time-step ordinary differential equation integrator.

Another method to solve the transient heat conduction equation is to transform the differential equation to an eigenvalue problem. The key feature of this approach is to express the time-dependent temperature field in a linear combination of expressions which have exponential time dependence. It can be shown that the expansion in time is exact and no errors involving time are introduced. The eigenvalue method has been used by researchers in structural mechanics[2] but has received little attention in the heat transfer community. Zhong presented a variational method for solving transient heat conduction problems[3] in which an approximate closed-form analytical solution is available. An eigenvalue method in finite difference scheme was applied to the transient heat conduction by Shih and Skladany[4]. Landry and Kaplan[5] performed a comparison of the computer time required for the eigenvalue method and the conventional methods. They pointed out that the eigenvalue method requires a large amount of computer time when the number of spatial meshes is large. Haji-Sheikh *et al.*[6,7] presented an analytical solution of the transient heat conduction equation in which the Galerkin method was used to calculate the eigenvalues but their form of solution was rather complicated because normalized eigenvectors were not used.

The present work extends the analytical procedure used[3] to solve the transient heat conduction problems using the finite element scheme. An analytical solution for the temperature at each finite element node is derived. Since the discrete analytical solution is given in closed form, it is convenient to extend the present method to treat fluid–solid and thermal–structure interaction problems.

To test the validity of the present method, three test examples which possess exact solutions are solved. The major portion of the computation time is used to compute the values of eigenvalues and their corresponding eigenvectors. These eigenvalues and eigenvectors are calculated numerically using readily available library eigensystem routines.

VARIATIONAL WEAK FORMULATION AND LAGRANGIAN EQUATION

Consider the problem of transient heat conduction governed by:

$$\frac{\partial}{\partial x_\alpha}\left(k\frac{\partial\theta}{\partial x_\alpha}\right)+W=\rho c\frac{\partial\theta}{\partial t}\qquad\text{in}\quad\Omega \tag{1}$$

The initial condition is:

$$\theta(x_\alpha,0)=\theta_0(x_\alpha)\qquad\text{in}\quad\Omega \tag{2}$$

and the boundary conditions are

$$\theta(x_\alpha,t)=\theta_w(x_\alpha,t)\qquad\text{in}\quad\Gamma_1 \tag{3}$$

$$k\frac{\partial\theta}{\partial n}(x_\alpha,t)=F(x_\alpha,t)\qquad\text{on}\quad\Gamma_2 \tag{4}$$

$$k\frac{\partial\theta}{\partial n}(x_\alpha,t)=-h(\theta-\theta_\infty)\qquad \text{on}\quad \Gamma_3 \tag{5}$$

$$\left.\begin{aligned}k\frac{\partial\theta}{\partial n}(x_\alpha,t)&=-h_r[4(\theta+T_0)-3(\bar\theta+T_0)]+F_{r\infty}\\ h_r&=\varepsilon\sigma(\bar\theta+T_0)^3\\ F_{r\infty}&=\varepsilon\sigma(\theta_\infty+T_0)^4\end{aligned}\right\}\text{on}\quad\Gamma_4 \tag{6}$$

Here W, k, ρ and c are functions of position only, and $\Gamma=\Sigma\Gamma_i$ is the surface bounding the closed region Ω; $\theta=T-T_0$ where T_0 is the initial uniform temperature or an arbitrary reference temperature; $\bar\theta$ is the trial surface temperature for iteration in linearized computation of the radiative boundary condition; ε is the emissivity and σ is the Stefan–Boltzmann constant.

Multiplying (1) by the variation of θ and integrating over the spatial domain Ω, one can obtain the variational formulation of the transient heat conduction equation[3,8]:

$$\delta E+\delta R=\delta A \tag{7}$$

where

$$E=\int_\Omega\left[\tfrac{1}{2}k\left(\frac{\partial\theta}{\partial x_\alpha}\right)^2-W\theta\right]\mathrm{d}\Omega \tag{8}$$

$$\delta R=\int_\Omega \rho c\frac{\partial\theta}{\partial t}\delta\theta\,\mathrm{d}\Omega \tag{9}$$

$$\delta A=\int_\Gamma k\frac{\partial\theta}{\partial n}\delta\theta\,\mathrm{d}\Gamma \tag{10}$$

If we introduce the boundary conditions (3) to (6) into (7), the variational formulation can be written as:

$$\delta E+\delta R+\delta E_w=\delta A_w \tag{11}$$

where

$$\delta E_w=\int_{\Gamma_3,\Gamma_4}(4h_r+h)\theta\,\delta\theta\,\mathrm{d}\Gamma \tag{12}$$

$$\delta A_w=\int_{\Gamma_2,\Gamma_3,\Gamma_4}(F+h\theta_\infty+3h_r\bar\theta-h_rT_0+F_{r\infty})\,\delta\theta\,\mathrm{d}\Gamma \tag{13}$$

We shall now construct the temperature field by means of a set n generalized coordinates $q_i(t)$, that is:

$$\theta=\theta(q_i,x_\alpha)\qquad i=1,2,\ldots,n \tag{14}$$

Using the generalized coordinates q_i, the variational formulation (11) leads to the Lagrangian equation:

$$\frac{\partial(E+E_w)}{\partial q_i}+\frac{\partial R}{\partial\dot q_i}=Q_i \tag{15}$$

where $\dot q_i=\mathrm{d}q_i/\mathrm{d}t$ and

$$R=\int_\Omega\tfrac{1}{2}\rho c\left(\frac{\partial\theta}{\partial t}\right)^2\mathrm{d}\Omega \tag{16}$$

$$E_w=\int_{\Gamma_3,\Gamma_4}\tfrac{1}{2}(4h_r+h)\theta^2\,\mathrm{d}\Gamma \tag{17}$$

$$Q_i=\int_{\Gamma_2,\Gamma_3,\Gamma_4}(F+h\theta_\infty+3h_r\bar{\theta}-h_rT_0+F_{r\infty})\frac{\partial\theta}{\partial q_i}\mathrm{d}\Gamma \tag{18}$$

DISCRETE FINITE ELEMENT CLOSED-FORM SOLUTION

The region Ω is divided into M elements and with N nodes. The discretization of the temperature field $\theta(x_\alpha,t)$ is carried out in terms of the generalized coordinates $q_i(t)$ using the finite element scheme.

Suppose that the local approximation for θ in an element is expressed by:

$$\theta^{(e)}=f_g^{(e)}\theta_g^{(e)}+f_l^{(e)}q_l^{(e)} \tag{19}$$

$$g=1,\ldots,G;\qquad l=1,\ldots,L$$

where $f_l^{(e)}(x_\alpha)$ is the local position function which is defined to have the value of unity at node l and zero at all the other nodes in a local element Ω_e; the generalized coordinate $q_l^{(e)}$ represents the unknown value of θ at node l; and $\theta_g^{(e)}$ is the known temperature at node g. L is the number of unknown nodes in an element and G is the number of nodes at which the value of θ is given. Throughout this paper, the unknown node is denoted by subscript l or m and the known node by g or h.

By inserting (19) into (8), (16), (17) and (18), the local functions $E^{(e)}$, $R^{(e)}$, $E_w^{(es)}$ and $Q_l^{(es)}$ in terms of the generalized coordinates $q_l^{(e)}$ can be obtained. The global functions E, R, E_w and Q_i can be evaluated by summing the contributions from the individual elements (see Appendix A for details)

$$E=\sum_{e=1}^{M}E^{(e)}=\tfrac{1}{2}a_{ij}q_iq_j+a_iq_i+a_0 \tag{20}$$

$$R=\sum_{e=1}^{M}R^{(e)}=\tfrac{1}{2}b_{ij}\dot{q}_i\dot{q}_j+b_i\dot{q}_i+b_0 \tag{21}$$

$$E_w=\sum_{es=1}^{M_3,M_4}E_w^{(es)}=\tfrac{1}{2}c_{ij}q_iq_j+c_iq_i+c_0 \tag{22}$$

$$Q_i=\sum_{es=1}^{M_2,M_3,M_4}Q_l^{(es)}\Delta_{li}^{(es)} \tag{23}$$

where $\Delta_{li}^{(es)}$ is the Boolean matrix, that is:

$$\Delta_{li}^{(es)}=\begin{cases}1 & l=i\\ 0 & l\neq i\end{cases} \tag{24}$$

By substituting (20) to (23) into (15), we obtain the following system of ordinary differential equations:

$$(a_{ij}+c_{ij})q_j+b_{ij}\dot{q}_j=f_i \tag{25}$$

where

$$f_i=Q_i-a_i-b_i-c_i \tag{26}$$

We seek a solution in the form:

$$q_j=\phi_j\mathrm{e}^{-\lambda t} \tag{27}$$

Direct substitution of the above form into the homogeneous part of (25) leads to:

$$(a_{ij}+c_{ij}-\lambda b_{ij})\phi_j=0 \tag{28}$$

which is a generalized eigenvalue problem. If there are n generalized coordinates q_i $(i=1,2,\ldots,n)$ in the system, there will be n eigenvalues $\lambda_s(s=1,2,\ldots,n)$ with n corresponding eigenvectors $\phi_i^{(s)}$. The matrices $[A+C]=(a_{ij}+c_{ij})_{n\times n}$ and $[B]=(b_{ij})_{n\times n}$ are both positive definite, therefore the eigenvalues λ_s are all real and positive.

The general solution to the homogeneous part of (25) can be expressed as a linear combination of $\phi_i^{(s)}\mathrm{e}^{-\lambda_s t}$,

$$q_i=\sum_{s=1}^{n} c^{(s)}\phi_i^{(s)}\mathrm{e}^{-\lambda_s t} \tag{29}$$

where $c^{(s)}$ $(s=1,2,\ldots,n)$ are arbitrary constants which will be determined later through normalization of the eigenvectors $\phi_i^{(s)}$.

To find a closed-form solution to (25), we introduce the principal coordinates p_s $(s=1,2,\ldots,n)$, which is defined by:

$$\{q\}=[Y]\{p\} \tag{30}$$

where $[Y]=(y_{is})_{n\times n}$ and $(y_{is})=c^{(s)}\phi_i^{(s)}$.

The Lagrangian equations (15) can be written in the following form:

$$\frac{\partial(E+E_w)}{\partial p_s}+\frac{\partial R}{\partial \dot{p}_s}=\pi_s \tag{31}$$

where

$$\pi_s=\sum_{i=1}^{n} c^{(s)}\phi_i^{(s)}Q_i \tag{32}$$

Using the following orthogonal relations,

$$(c^{(s)}\phi^{(s)})^{\mathrm{T}}[A+C]\{c^{(r)}\phi^{(r)}\}=0, \qquad r\neq s \tag{33a}$$

$$\{c^{(s)}\phi^{(s)}\}^{\mathrm{T}}[B]\{c^{(r)}\phi^{(r)}\}=0, \qquad r\neq s \tag{33b}$$

the functions E, E_w, and R can be written in terms of the principal coordinates:

$$E+E_w=\tfrac{1}{2}\sum_{s=1}^{n}\alpha_s p_s^2+\sum_{s=1}^{n} d_s p_s+a_0+c_0 \tag{34}$$

$$R=\tfrac{1}{2}\sum_{s=1}^{n}\beta_s \dot{p}_s^2+\sum_{s=1}^{n} e_s \dot{p}_s+b_0 \tag{35}$$

where

$$\alpha_s=\{c^{(s)}\phi^{(s)}\}^{\mathrm{T}}[A+C]\{c^{(s)}\phi^{(s)}\} \tag{36}$$

$$\beta_s=\{c^{(s)}\phi^{(s)}\}^{\mathrm{T}}[B]\{c^{(s)}\phi^{(s)}\} \tag{37}$$

$$d_s=\sum_{i=1}^{n}(a_i+c_i)c^{(s)}\phi_i^{(s)} \tag{38}$$

$$e_s=\sum_{i=1}^{n} b_i c^{(s)}\phi_i^{(s)} \tag{39}$$

$$\alpha_s=\lambda_s\beta_s \tag{40}$$

By substituting (34) and (35) into (31), we have

$$\alpha_s p_s + \beta_s \dot{p}_s = \pi_s - d_s - e_s \tag{41}$$

We can choose as normalization by setting:

$$\beta_s = 1 \tag{42}$$

and the solution of (41) is:

$$p_s(t) = \mathrm{e}^{-\lambda_s(t-t_0)}\left[\int_{t_0}^{t} (\pi_s - d_s - e_s)\,\mathrm{e}^{\lambda_s(t-t_0)}\,\mathrm{d}t + p_s(t_0)\right] \tag{43}$$

where $p_s(t_0)$ is the initial value of the principal coordinate p_s. A similar form of (43) is given in Reference 3 except that (43) is directly applicable when the spatial domain is discretized by finite element approximations.

If $(\pi_s - d_s - e_s)$ is constant, i.e., the boundary conditions are independent of time, (43) can be integrated to give:

$$p_s(t) = \frac{\pi_s - d_s - e_s}{\lambda_s}(1 - \mathrm{e}^{-\lambda_s(t-t_0)}) + p_s(t_0)\,\mathrm{e}^{-\lambda_s(t-t_0)}, \qquad \lambda_s \neq 0 \tag{44}$$

$$p_s(t) = (\pi_s - d_s - e_s)(t - t_0) + p_s(t_0), \qquad \lambda_s = 0 \tag{45}$$

The initial values of p_s in (43) can be determined from (30) by using the initial value of $q_i(t_0)$, that is:

$$\{p(t_0)\} = [Y]^{-1}\{q(t_0)\} \tag{46}$$

where $[Y]^{-1}$ can be calculated by following[3]:

$$[Y]^{-1} = [Y]^{\mathrm{T}}[\mathrm{B}] \qquad \beta_s = 1 \tag{47}$$

From the above, we can see that the major computational effort for the eigenvalue method involves the calculation of the eigenvalues and the corresponding eigenvectors based on (28). The eigenvalues and eigenvectors can be obtained analytically if n is small. For large n, they can be carried out using EISPACK[9] or other library routines.

Once the eigenvalues and eigenvectors are known, the temperature field at any given time can be obtained in closed form through (30) and (43). It is important to note that for a given geometry, the eigenvalues and eigenvectors need to be solved only once for the entire time domain except when the convective or radiative heat transfer coefficients are time-dependent or temperature-dependent.

COMPUTATIONAL EXAMPLES

Square domain with mixed boundary conditions

In order to assess the accuracy of the eigenvalue method, we first consider a square domain with mixed boundary conditions as shown in *Figure 1*. The governing equation is:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{\partial\theta}{\partial t} \tag{48}$$

with boundary conditions

$$\theta(0, y, t) = 1, \qquad \theta(1, y, t) = 0 \tag{49}$$

$$\frac{\partial\theta}{\partial y}(x, 0, t) = 0, \qquad \frac{\partial\theta}{\partial y}(x, 1, t) = 0$$

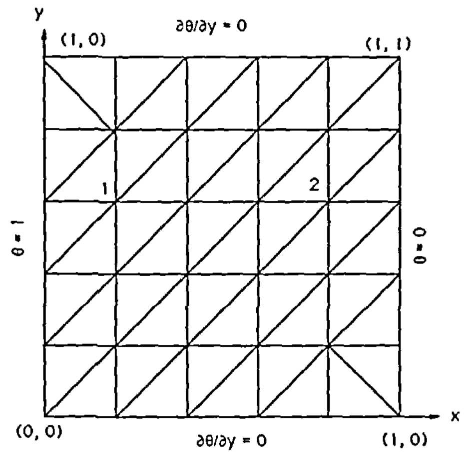

Figure 1 Square domain with mixed boundary conditions

and initial condition

$$\theta(x,y,0)=0 \tag{50}$$

The exact solution is[10]:

$$\theta(x,y,t)=1-x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(1-x)n\pi \tag{51}$$

Numerical solutions were obtained by using 50 linear triangular elements with 36 nodes (*Figure 1*) and 200 linear triangular elements with 121 nodes. The temperatures at locations 1 and 2 shown in *Figure 1* are presented in *Table 1* where they are compared with the exact solution. It can be seen that the agreement is excellent and the errors decrease as time increases. The error is larger at smaller times due to the initial step increase in temperature at the boundary. In addition, the accuracy of the numerical solutions using the coarser grid is comparable to that of the finer grid.

The results given in *Table 1* are calculated when the full set of eigenvalue and eigenvectors were used. For the coarser grid, there are 24 eigenvalues and eigenvectors, whereas for the finer grid, there are 99. Since the computation of a partial set of eigenvalues and eigenvectors takes much less computer time than the full set, we look into the numerical results based on two partial sets. The first set, called the direct set, groups eigenvalues and eigenvectors based on the magnitude of the eigenvalues starting with the smallest eigenvalue. The second set, called the

Table 1 Comparison between present method (50 and 200 elements) and exact solution (example 1, mixed boundary condition)

	q_1			q_2		
Time	Exact	50 E	200 E	Exact	50 E	200 E
0.1	0.6605	0.6662	0.6581	0.0605	0.0719	0.0677
0.2	0.7480	0.7529	0.7498	0.1480	0.1528	0.1497
0.3	0.7806	0.7830	0.7821	0.1806	0.1830	0.1818
0.4	0.7928	0.7939	0.7940	0.1928	0.1939	0.1937
0.5	0.7973	0.7978	0.7984	0.1973	0.1978	0.1981

decomposed set, groups the contributions to the numerical result by separating terms with or without the exponential time dependence.

Direct set: When the boundary condition is independent of time, the contribution of each eigenmode to the temperature at node i is given by combining (30) and (44),

$$q_i^{(s)} = \sum_{j=1}^{n} \{c^{(s)}\}^2 \phi_i^{(s)} \phi_j^{(s)} (Q_j - a_j - b_j - c_j)(1 - e^{-\lambda_s t})/\lambda_s \tag{52}$$

The cumulative contribution of the eigenmodes to the temperatures at locations 1 and 2 are plotted *versus* the number of eigenmodes in *Figure 2* for the case of 50 elements with 24 generalized coordinates and in *Figure 3* for the case of 200 elements with 99 generalized coordinates. It can be seen that in order to have reasonably accurate numerical results, approximately half of the eigenmodes are needed for both cases.

Decomposed set: The solution given by (30) and (44) can be written as:

$$q_i = q_i^* - \sum_{s=1}^{n} H_i^{(s)}(t) \tag{53}$$

where

$$q_i^* = \sum_{s=1}^{n} c^{(s)} \phi_i^{(s)} \frac{\pi_s - d_s - e_s}{\lambda_s} \tag{54}$$

and

$$H_i^{(s)}(t) = c^{(s)} \phi_i^{(s)} \frac{\pi_s - d_s - e_s}{\lambda_s} e^{-\lambda_s t} \tag{55}$$

In (53), q_i^* is in effect the steady-state solution, and $H_i^{(s)}(t)$ are the unsteady mode terms. The values for $H_i^{(s)}$ are listed in *Table 2* and their cumulative contribution to the temperatures at locations 1 and 2 are also shown. It is clear that 3 or less eigenmodes are needed if the steady-state solution q_i^* is known *a priori*. If q_i^* is not known, rather than determining it through (54), it is easier to solve (25) by dropping the time-dependent term.

The temperature at each node can be written in the form of (53). For example, the temperatures at locations 1 and 2 can be expressed as follows:

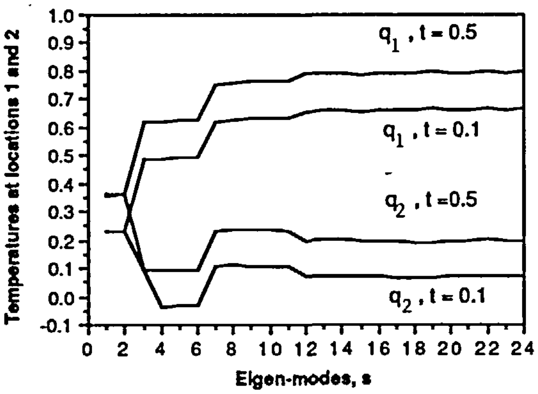

Figure 2 Cumulative contribution of eigenmodes to temperatures at locations 1 and 2—50 elements with 24 generalized coordinates

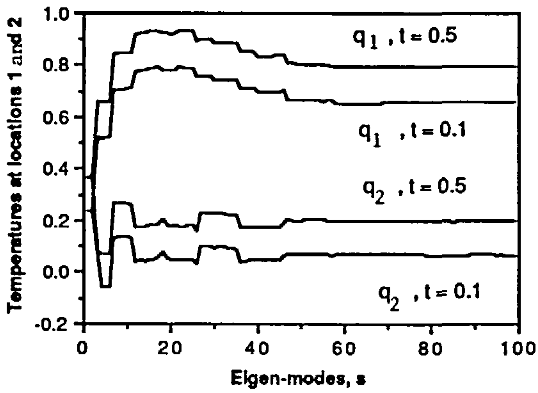

Figure 3 Cumulative contribution of eigenmodes to temperatures at locations 1 and 2—200 elements with 99 generalized coordinates

Table 2 Values of eigenmodes and their contribution to the temperatures at locations 1 and 2 (example 1)

		$t=0.1$				$t=0.5$			
s	λ_s	$H_1^{(s)}$	q_1	$H_2^{(s)}$	q_2	$H_1^{(s)}$	q_1	$H_2^{(s)}$	q_2
				24 generalized coordinates					
1	10.19	0.1305	0.6695	0.1308	0.0692	0.0022	0.7978	0.0022	0.1978
2	21.75	0.0004	0.6691	0.0003	0.0689	0.0000	0.7978	0.0000	0.1978
3	44.77	0.0029	0.6662	−0.0030	0.0719				
4	57.73	0.0000	0.6662	0.0000	0.0719				
				99 generalized coordinates					
1	9.93	0.1377	0.6632	0.1379	0.0628	0.0026	0.7984	0.0026	0.1981
2	20.21	0.0001	0.6631	0.0001	0.0627	0.0000	0.7984	0.0000	0.1981
3	40.77	0.0049	0.6581	−0.0050	0.0677				
4	51.27	0.0000	0.6581	0.0000	0.0677				

With 50 elements

$$\theta_1(t)=0.8-0.3617\,e^{-10.19t}-0.00374\,e^{-21.75t}-0.2561\,e^{-44.77t} \tag{56a}$$

$$\theta_2(t)=0.2-0.3626\,e^{-10.19t}-0.00265\,e^{-21.75t}+0.2664\,e^{-44.77t} \tag{56b}$$

With 200 elements

$$\theta_1(t)=0.8-0.3718\,e^{-9.93t}-0.00112\,e^{-20.21t}-0.2907\,e^{-40.77t} \tag{57a}$$

$$\theta_2(t)=0.2-0.3721\,e^{-9.93t}-0.00103\,e^{-20.21t}+0.2945\,e^{-40.77t} \tag{57b}$$

when $0.1 \leqslant t \leqslant 0.5$. When $t>0.5$, only the first unsteady term needs to be included.

Long-time solution with time-dependent boundary conditions

The long-time solution with time-dependent boundary conditions will be considered for a one-dimensional plane wall with constant properties k, ρ, and c. The ambient temperature on the left-hand side is a time-dependent periodic function and the ambient temperature on the right-hand side is kept constant as shown in *Figure 4*.

For large time, the exact solution of the surface temperature θ_L at the right-hand side is:

$$\theta_L=\theta_{\infty,L}+\frac{A\sin(\bar{\omega}F_0)-\theta_{\infty,L}}{2+B_i}-A\bar{\omega}\sum_{\kappa=1}^{\infty}\frac{[B_i^2+(\lambda_\kappa L)^2\sin(\lambda_\kappa L)[(\lambda_\kappa L)^2\cos(\bar{\omega}F_0)+\bar{\omega}\sin(\bar{\omega}F_0)]}{(\lambda_\kappa L)[(\lambda_\kappa L)^2+2B_i+B_i^2][(\lambda_\kappa L)^4+\bar{\omega}^2]} \tag{58}$$

where

$$F_0=\frac{k}{\rho cL^2}t, \qquad B_i=\frac{hL}{k} \qquad \bar{\omega}=\frac{\omega}{k/\rho cL^2}$$

and $(\lambda_\kappa L)$ are the positive roots of:

$$\tan(\lambda_\kappa L)=\frac{2B_i(\lambda_\kappa L)}{(\lambda_\kappa L)^2-B_i^2} \tag{59}$$

We have solved this problem using two and eight one-dimensional linear elements. However, to illustrate the eigenvalue method, we will only give the details for the case of two linear elements

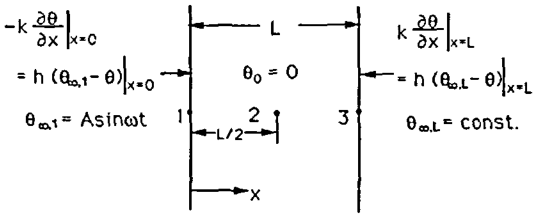

Figure 4 One-dimensional plate with a time-dependent boundary condition

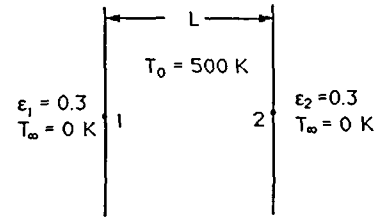

Figure 5 One-dimensional plate with radiative boundary condition

with three nodes. The matrices [A], [B], [C] and Q_i are given as follows:

$$[A]=\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\frac{2k}{L} \qquad [B]=\begin{bmatrix}1 & \frac{1}{2} & 0\\ \frac{1}{2} & 2 & \frac{1}{2}\\ 0 & \frac{1}{2} & 1\end{bmatrix}\frac{\rho cL}{6} \qquad [C]=\begin{bmatrix}h & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & h\end{bmatrix} \tag{60}$$

$$Q_1=h\theta_{\infty,1}=hA\sin\omega t \qquad Q_2=0 \qquad Q_3=h\theta_{\infty,L} \tag{61}$$

The solution of the generalized eigenvalue problem, (28) yields the following eigenvalues and eigenvectors

$$\lambda_1=2.6709k/\rho cL^2 \qquad \lambda_2=21.7330k/\rho cL^2 \qquad \lambda_3=58.3065k/\rho cL^2 \tag{62}$$

$$\{\phi^{(1)}\}=\begin{bmatrix}-0.8190\\ -1.1707\\ -0.8190\end{bmatrix} \qquad \{\phi^{(2)}\}=\begin{bmatrix}1.734\\ -1.6216\\ 1.8246\end{bmatrix} \qquad \{\phi^{(3)}\}=\begin{bmatrix}1.8246\\ -1.6216\\ 1.8246\end{bmatrix} \tag{63}$$

Here the eigenvectors were normalized, that is:

$$c^{(s)}=(1/\rho cL)^{1/2} \qquad s=1,2,3 \tag{64}$$

Since there are no known boundary nodes in the problem, d_s as defined by (38) and e_s as defined by (39) are both zero. The π_s are given by substituting (61) into (32)

$$\pi_s=c^{(s)}\phi_1^{(s)}hA\sin\omega t+c^{(s)}\phi_3^{(s)}h\theta_{\infty,L} \tag{65}$$

Assuming the initial temperature θ_0 is zero, it follows that $q_i(0)=0$ and $p_s(0)=0$. The solution of the temperature field can be obtained using (30) and (43):

$$p_s(t)=c^{(s)}\phi_1^{(s)}hA\frac{\lambda_s}{\omega^2+\lambda_s^2}\left(\sin\omega t-\frac{\omega}{\lambda_s}\cos\omega t+\frac{\omega}{\lambda_s}e^{-\lambda_s t}\right)+c^{(s)}\phi_3^{(s)}h\theta_{\infty,L}\frac{(1-e^{-\lambda_s t})}{\lambda_s} \tag{66}$$

$$q_i(t)=\sum_{s=1}^{3}c^{(s)}\phi_i^{(s)}p_s(t) \qquad i=1,2,3 \tag{67}$$

The temperature at the right boundary (node 3) can be written as:

$$\theta_L=q_3=B_i\left[A\left[\sin\bar{\omega}F_0\sum_{s=1}^{3}\phi_1^{(s)}\phi_3^{(s)}\frac{\bar{\lambda}_s}{\bar{\omega}^2+\bar{\lambda}_s^2}-\cos\bar{\omega}F_0\sum_{s=1}^{3}\phi_1^{(s)}\phi_3^{(s)}\frac{\bar{\omega}}{\bar{\omega}^2+\bar{\lambda}_s^2}+\right.\right.$$
$$\left.\left.\sum_{s=1}^{3}\phi_1^{(s)}\phi_3^{(s)}\frac{\bar{\omega}}{\bar{\omega}^2+\bar{\lambda}_s^2}+\sum_{s=1}^{3}\phi_1^{(s)}\phi_3^{(s)}\frac{\bar{\omega}}{\bar{\omega}^2+\bar{\lambda}_s^2}e^{-\bar{\lambda}_s F_0}\right]+\theta_{\infty,L}\sum_{s=1}^{3}\phi_3^{(s)2}\frac{1-e^{-\bar{\lambda}_s F_0}}{\bar{\lambda}_s}\right] \tag{68}$$

Table 3 Comparison between present method and exact long-time solution (time-dependent boundary conditions)

			θ_L/A		
			Present method		Exact solution
Cycle	t	$\theta_{\infty,1}/A$	Two elements	Eight elements	
1	0	0.00	0.0000	0.0000	
	6	1.00	−0.5210	−0.5170	
	12	0.00	−0.5245	−0.5311	
	18	−1.00	−0.8204	−0.8196	
2	24	0.00	−0.9053	−0.8962	
	30	1.00	−0.6244	−0.6244	
	36	0.00	−0.5421	−0.5510	
	42	−1.00	−0.8234	−0.8233	
3	48	0.00	−0.9058	−0.8968	
	54	1.00	−0.6245	−0.6246	
	60	0.00	−0.5421	−0.5510	
	66	−1.00	−0.8234	−0.8233	
4	72	0.00	−0.9058	−0.8968	−0.8963
	78	1.00	−0.6245	−0.6246	−0.6247
	84	0.00	−0.5421	−0.5510	−0.5517
	90	−1.00	−0.8234	−0.8233	−0.8233

where

$$\bar{\lambda}_s = \lambda_s \bigg/ \frac{k}{\rho c L^2} \tag{69}$$

Table 3 gives a comparison between the present method using two and eight elements and the long-time exact solution (58) for the case of $k/\rho cL^2 = 0.1107\,(\mathrm{h})^{-1}$, $\omega = \pi/12\,(\mathrm{h})^{-1}$, $\theta_{\infty,L}/A = -1$ and $B_i = 1.622$. Good agreement with the exact long-time solution is achieved after the third cycle.

Example with radiative boundary conditon

As a final example, we consider a simple one-dimensional problem with radiative boundary conditions on both sides as shown in *Figure 5*. A single linear element is chosen and the matrices $[A]$, $[B]$, and $[C]$ are given as follows:

$$[A] = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [B] = \frac{\rho c L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad [C] = 4\begin{bmatrix} h_r(\bar{\theta}_1) & 0 \\ 0 & h_r(\bar{\theta}_2) \end{bmatrix} \tag{70}$$

and the generalized Q_i are:

$$Q_1 = h_r(\bar{\theta}_1)(3\bar{\theta}_1 - T_0) \qquad Q_2 = h_r(\bar{\theta}_2)(3\bar{\theta}_2 - T_0) \tag{71}$$

For a plate with very high thermal conductivity, a uniform temperature can be assumed, the exact solution is:

$$\frac{T}{T_0} = \left(1 + \frac{6\varepsilon\sigma T_0^3 t}{\rho c L}\right)^{-1/3} \tag{72}$$

Since only one linear element was used and the two boundary temperatures are the same, the spatial temperature variation within the solid is ignored. Thus, the numerical solution should be compared with the lumped capacity solution given by (72). *Table 4* describes the iterative process at $t = 1$ h, where $k/L = 5\ \mathrm{W/m^2.K}$ and $\rho cL = 10\ \mathrm{W.h/m^2.K}$. The initial condition is given by the results at the previous time at $t = 0.9$ h. It is shown that the convergence is very fast.

Table 4 Iterative process for radiative boundary condition

	Iteration		
	1	2	3
T_1	387.6186205	380.235582	380.2350751
T_2	$=T_1$	$=T_1$	$=T_1$
λ_1	0.7924743537	0.7480482081	0.7480452165
λ_2	8.377423064	8.244144628	8.244135653
$c^{(1)}\phi_1^{(1)}$	0.316227766	0.316227766	0.316227766
$c^{(2)}\phi_1^{(2)}$	0.5477225576	0.5477225577	0.5477225577
$c^{(1)}\phi_2^{(1)}$	0.316227766	0.316227766	0.316227766
$c^{(2)}\phi_2^{(2)}$	−0.5477225576	−0.5477225577	−0.5477225577
T_1	380.235582	380.2350751	380.2350761
T_2	$=T_1$	$=T_1$	$=T_1$

Table 5 Comparison of surface temperature between present method and exact solution (radiative boundary condition)

t(h)	Present method	Exact method
0.1	480.44	480.38
0.2	463.61	463.53
0.3	448.91	448.81
0.4	435.91	435.80
0.5	424.29	424.18
0.6	413.82	413.70
0.7	404.30	404.19
0.8	395.61	395.50
0.9	387.62	387.51
1.0	380.24	380.13

The surface temperature obtained by the present method is compared with the exact solution in *Table 5* and they are in excellent agreement.

CONCLUSIONS

A finite-element eigenvalue method was presented for solving transient heat conduction problems. The major advantage of this method over other conventional numerical methods is that there is no time-step restriction. When a large number of finite element nodes is needed. Reasonably accurate results can be obtained by using only a partial set of eigenvalues and eigenvectors, especially when the problem at hand has a steady-state solution which is known or can be determined by some other means. A closed-form expression for the temperature field is available which is very useful for coupled thermal–structure interaction problems. The results obtained using the present method were in excellent agreement with the exact solutions for three test problems.

This method can be extended to solve various types of parabolic equations such as the transient heat conduction problem with phase change, and boundary-layer type equations.

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APPENDIX A

The coefficients a_{ij}, a_i and a_0 in (20) are given as follows:

$$a_{ij}=a_{ji}=\sum_{e=1}^{M} a_{lm}^{(e)}\Delta_{li}^{(e)}\Delta_{mj}^{(e)} \tag{A1}$$

$$a_i=\sum_{e=1}^{M} a_l^{(e)}\Delta_{li}^{(e)} \tag{A2}$$

$$a_0=\sum_{e=1}^{M} a_0^{(e)} \tag{A3}$$

where

$$a_{lm}^{(e)}=a_{ml}^{(e)}=\int_{\Omega^{(e)}} k^{(e)}\frac{\partial f_l^{(e)}}{\partial x_\alpha}\frac{\partial f_m^{(e)}}{\partial x_\alpha}\mathrm{d}\Omega \tag{A4}$$

$$a_l^{(e)}=\left[\int_{\Omega^{(e)}} k^{(e)}\frac{\partial f_g^{(e)}}{\partial x_\alpha}\frac{\partial f_l^{(e)}}{\partial x_\alpha}\mathrm{d}\Omega\right]\theta_g^{(e)}-\int_{\Omega^{(e)}} W^{(e)}f_l^{(e)}\,\mathrm{d}\Omega \tag{A5}$$

$$a_0^{(e)}=\tfrac{1}{2}\left[\int_{\Omega^{(e)}} k^{(e)}\frac{\partial f_g^{(e)}}{\partial x_\alpha}\frac{\partial f_h^{(e)}}{\partial x_\alpha}\mathrm{d}\Omega\right]\theta_g^{(e)}\theta_h^{(e)}-\left[\int_{\Omega^{(e)}} W^{(e)}f_g^{(e)}\,\mathrm{d}\Omega\right]\theta_g^{(e)} \tag{A6}$$

The coefficients b_{ij}, b_i and b_0 in (21) are given as follows:

$$b_{ij}=b_{ji}=\sum_{e=1}^{M} b_{lm}^{(e)}\Delta_{li}^{(e)}\Delta_{mj}^{(e)} \tag{A7}$$

$$b_i=\sum_{e=1}^{M} b_l^{(e)}\Delta_{li}^{(e)} \tag{A8}$$

$$b_0=\sum_{e=1}^{M} b_0^{(e)} \tag{A9}$$

and

$$b_{lm}^{(e)} = b_{ml}^{(e)} = \int_{\Omega^{(e)}} \rho^{(e)} c^{(e)} f_l^{(e)} f_m^{(e)} \, d\Omega \tag{A10}$$

$$b_l^{(e)} = \left[\int_{\Omega^{(e)}} \rho^{(e)} c^{(e)} f_g^{(e)} f_l^{(e)} \, d\Omega \right] \dot{\theta}_g^{(e)} \tag{A11}$$

$$b_0^{(e)} = \tfrac{1}{2} \left[\int_{\Omega^{(e)}} \rho^{(e)} c^{(e)} f_g^{(e)} f_h^{(e)} \, d\Omega \right] \dot{\theta}_g^{(e)} \dot{\theta}_h^{(e)} \tag{A12}$$

The coefficients c_{ij}, c_i and c_0 in (22) are given as follows:

$$c_{ij} = c_{ji} = \sum_{es=1}^{M_3, M_4} c_{lm}^{(es)} \Delta_{li}^{(es)} \Delta_{mj}^{(es)} \tag{A13}$$

$$c_i = \sum_{es=1}^{M_3, M_4} c_l^{(es)} \Delta_{li}^{(es)} \tag{A14}$$

$$c_0 = \sum_{es=1}^{M_3, M_4} c_0^{(es)} \tag{A15}$$

where

$$c_{lm}^{(es)} = c_{ml}^{(es)} = \int_{\Gamma_3, \Gamma_4} (4h_r^{(es)} + h^{(es)}) \bar{f}_l^{(es)} \bar{f}_m^{(es)} \, d\Gamma \tag{A16}$$

$$c_l^{(es)} = \left[\int_{\Gamma_3, \Gamma_4} (4h_r^{(es)} + h^{(es)}) \bar{f}_g^{(es)} f_l^{(es)} \, d\Gamma \right] \theta_g^{(es)} \tag{A17}$$

$$c_0^{(es)} = \tfrac{1}{2} \left[\int_{\Gamma_3, \Gamma_4} (4h_r^{(es)} + h^{(es)}) \bar{f}_g^{(es)} \bar{f}_h^{(es)} \, d\Gamma \right] \theta_g^{(es)} \theta_h^{(es)} \tag{A18}$$

The contribution to Q_i by the individual element $Q_l^{(es)}$ given in (23) is:

$$Q_l^{(es)} = \int_{\Gamma_2, \Gamma_3, \Gamma_4} (F^{(es)} + h^{(es)} \theta_\infty^{(es)} + 3h_r^{(es)} \bar{\theta}^{(es)} - h_r^{(es)} T_0 + F_{r\infty}^{(es)}) \bar{f}_l^{(es)} \, d\Gamma \tag{A19}$$

In Appendix B, the coefficients of $a_{lm}^{(e)}$. $b_{lm}^{(e)}$, $c_{lm}^{(es)}$, $a_l^{(e)}$, $b_l^{(e)}$, $c_l^{(es)}$, $a_0^{(e)}$, $b_0^{(e)}$, $c_0^{(es)}$ and $Q_l^{(es)}$ for plane and axisymmetric linear triangular elements are listed in *Tables B1*, *B2* and *B3*.

APPENDIX B

Coefficients of $a_{lm}^{(e)}$, $b_{lm}^{(e)}$, $c_{lm}^{(es)}$, $a_l^{(e)}$, $b_l^{(e)}$, $c_l^{(es)}$, $a_0^{(e)}$, $b_0^{(e)}$, $c_0^{(es)}$, *and* $Q_l^{(es)}$ *for planar and axisymmetric linear triangular elements*

We consider only two-dimensional systems. For axisymmetric systems, the axial and radial coordinates are x and r, respectively. For planar systems, r is the same as the y coordinate. We assume the three nodes of a triangular element to be (I, J, K) and for the case of a boundary element, we choose nodes J and K to be at the boundary. An element can have one, two or three degrees of freedom corresponding to the number of generalized coordinates in the element. The coefficients $a_{lm}^{(e)}$, $b_{lm}^{(e)}$, $c_{lm}^{(es)}$, $a_l^{(e)}$, $b_l^{(e)}$, $c_l^{(es)}$, $a_0^{(e)}$, $b_0^{(e)}$, $c_0^{(es)}$ and $Q_l^{(es)}$ are listed in the *Tables B1*, *B2* and *B3*. The various functions in these Tables are defined as follows:

$$A12(\xi_l, \xi_m, \eta_l, \eta_m) = (\xi_l . \xi_m + \eta_l . \eta_m) \cdot \frac{k}{12\Delta_e} (\gamma_I + \gamma_J + \gamma_K)$$

$$B11(\gamma_l,\gamma_m,\gamma_n)=\frac{\rho c\Delta_e}{30}(3\gamma_l+\gamma_m+\gamma_n)$$

$$B12(\gamma_l,\gamma_m,\gamma_n)=\frac{\rho c\Delta_e}{60}(2\gamma_l+2\gamma_m+\gamma_n)$$

$$C22(\gamma_m,\gamma_n)=\frac{\bar{h}s_I}{4}\left(\gamma_m+\frac{\gamma_n}{3}\right)$$

$$C23=\frac{\bar{h}s_I}{12}(\gamma_J+\gamma_K)$$

$$Q2(\gamma_m,\gamma_n)=\frac{\bar{F}s_I}{3}\left(\gamma_m+\frac{\gamma_n}{2}\right)$$

$$A1(\gamma_l,\gamma_m,\gamma_n)=-W\cdot\frac{\Delta_e}{12}(2\gamma_l,\gamma_m+\gamma_n)$$

In the above functions, for axisymmetric problems, $\gamma=r$ which is the radial distance of a node from the line of symmetry; for planar problems, $\gamma=1$. The variables ξ_l, η_l, for the triangular element with three nodes I, J, K are given by:

$$\xi_I=r_J-r_K \qquad \xi_J=r_K-r_I \qquad \xi_K=r_I-r_J$$

$$\eta_I=x_K-x_J \qquad \eta_J=x_I-x_K \qquad \eta_K=x_J-x_I$$

$$\Delta_e=\tfrac{1}{2}(\xi_I\eta_J-\xi_J\eta_I)$$

$$s_I=[(x_J-x_K)^2+(r_J-r_K)^2]^{1/2}$$

$$\bar{h}=h+4h_r \qquad h_r=\varepsilon\sigma(\bar{\theta}+T_0)^3$$

$$\bar{F}=F+h\theta_\infty+3h_r\bar{\theta}-h_rT_0+Fr_\infty$$

$$F_{r\infty}=\varepsilon\sigma(\theta_\infty+T_0)^4$$

Table B1 Coefficients of $a_{lm}^{(e)}$, $a_l^{(e)}$, and $a_0^{(e)}$ for an element with nodes I, J, K

Coeff.	3 freedoms	2 freedoms: θ_J known	2 freedoms: θ_K known	1 freedom: θ_J,θ_K known
$a_{II}^{(e)}$	$A12(\xi_I,\xi_I,\eta_I,\eta_I)$	$A12(\xi_I,\xi_I,\eta_I,\eta_I)$	$A12(\xi_I,\xi_I,\eta_I,\eta_I)$	$A12(\xi_I,\xi_I,\eta_I,\eta_I)$
$a_{JJ}^{(e)}$	$A12(\xi_J,\xi_J,\eta_J,\eta_J)$	0.0	$A12(\xi_J,\xi_J,\eta_J,\eta_J)$	0.0
$a_{KK}^{(e)}$	$A12(\xi_K,\xi_K,\eta_K,\eta_K)$	$A12(\xi_K,\xi_K,\eta_K,\eta_K)$	0.0	0.0
$a_{IJ}^{(e)}$	$A12(\xi_I,\xi_J,\eta_I,\eta_J)$	0.0	$A12(\xi_I,\xi_J,\eta_I,\eta_J)$	0.0
$a_{IK}^{(e)}$	$A12(\xi_I,\xi_K,\eta_I,\eta_K)$	$A12(\xi_I,\xi_K,\eta_I,\eta_K)$	0.0	0.0
$a_{JK}^{(e)}$	$A12(\xi_J,\xi_K,\eta_J,\eta_K)$	0.0	0.0	0.0
$a_I^{(e)}$	$A1(\gamma_I,\gamma_J,\gamma_K)$	$A12(\xi_I,\xi_J,\eta_I,\eta_J)\theta_J$ $+A1(\gamma_I,\gamma_J,\gamma_K)$	$A12(\xi_I,\xi_K,\eta_I,\eta_K)\theta_K$ $+A1(\gamma_I,\gamma_J,\gamma_K)$	$A12(\xi_I,\xi_J,\eta_I,\eta_J)\theta_J$ $+A12(\xi_I,\xi_K,\eta_I,\eta_K)\theta_K$ $+A1(\gamma_I,\gamma_J,\gamma_K)$
$a_J^{(e)}$	$A1(\gamma_J,\gamma_K,\gamma_I)$	0.0	$A12(\xi_J,\xi_K,\eta_J,\eta_K)\theta_K$ $+A1(\gamma_J,\gamma_K,\gamma_I)$	0.0
$a_K^{(e)}$	$A1(\gamma_K,\gamma_I,\gamma_J)$	$A12(\xi_J,\xi_K,\eta_J,\eta_K)\theta_J$ $+A1(\gamma_K,\gamma_I,\gamma_J)$	0.0	0.0
$a_0^{(e)}$	0.0	$A12(\xi_J,\xi_J,\eta_J,\eta_J)\theta_J^2$ $+A1(\gamma_J,\gamma_K,\gamma_I)\theta_J$	$A12(\xi_K,\xi_K,\eta_K,\eta_K)\theta_K^2$ $+A1(\gamma_K,\gamma_I,\gamma_J)\theta_K$	$A12(\xi_J,\xi_J,\eta_J,\eta_J)\theta_J^2$ $+2A12(\xi_J,\xi_K,\eta_J,\eta_K)\theta_J\theta_K$ $+A12(\xi_K,\xi_K,\eta_K,\eta_K)\theta_K^2$ $+A1(\gamma_J,\gamma_K,\gamma_I)\theta_J$ $+A1(\gamma_K,\gamma_I,\gamma_J)\theta_K$

Table B2 Coefficients of $b_{lm}^{(e)}$, $b_l^{(e)}$ and $b_0^{(e)}$ for a element with nodes I, J and K

Coeff.	3 freedoms	2 freedoms: θ_J known	2 freedoms: θ_K known	1 freedom: θ_J, θ_K known
$b_{II}^{(e)}$	$B11(\gamma_I,\gamma_J,\gamma_K)$	$B11(\gamma_I,\gamma_J,\gamma_K)$	$B11(\gamma_I,\gamma_J,\gamma_K)$	$B11(\gamma_I,\gamma_J,\gamma_K)$
$b_{JJ}^{(e)}$	$B11(\gamma_J,\gamma_K,\gamma_I)$	0.0	$B11(\gamma_J,\gamma_K,\gamma_I)$	0.0
$b_{KK}^{(e)}$	$B11(\gamma_K,\gamma_I,\gamma_J)$	$B11(\gamma_K,\gamma_I,\gamma_J)$	0.0	0.0
$b_{IJ}^{(e)}$	$B12(\gamma_I,\gamma_J,\gamma_K)$	0.0	$B12(\gamma_I,\gamma_J,\gamma_K)$	0.0
$b_{IK}^{(e)}$	$B12(\gamma_I,\gamma_K,\gamma_J)$	$B12(\gamma_I,\gamma_K,\gamma_J)$	0.0	0.0
$b_{JK}^{(e)}$	$B12(\gamma_J,\gamma_K,\gamma_I)$	0.0	0.0	0.0
$b_I^{(e)}$	0.0	$B12(\gamma_I,\gamma_J,\gamma_K)\dot{\theta}_J$	$B12(\gamma_I,\gamma_K,\gamma_J)\dot{\theta}_K$	$B12(\gamma_I,\gamma_J,\gamma_K)\dot{\theta}_J$ $+B12(\gamma_I,\gamma_K,\gamma_J)\dot{\theta}_K$
$b_J^{(e)}$	0.0	0.0	$B12(\gamma_J,\gamma_K,\gamma_I)\dot{\theta}_K$	0.0
$b_K^{(e)}$	0.0	$B12(\gamma_J,\gamma_K,\gamma_I)\dot{\theta}_J$	0.0	0.0
$b_0^{(e)}$	0.0	$B11(\gamma_J,\gamma_K,\gamma_I)\dot{\theta}_J$	$B11(\gamma_K,\gamma_I,\gamma_J)\dot{\theta}_K^2$	$B11(\gamma_J,\gamma_K,\gamma_I)\dot{\theta}_J^2$ $+2B12(\gamma_J,\gamma_K,\gamma_I)\dot{\theta}_J\dot{\theta}_K$ $+B11(\gamma_K,\gamma_I,\gamma_J)\dot{\theta}_K^2$

Table B3 Coefficients of $c_{lm}^{(es)}$, $c_l^{(es)}$, $c_0^{(es)}$ and $Q_l^{(es)}$ for boundary nodes J and K

Coefficients	2 freedoms	1 freedom: θ_J known	1 freedom: θ_K known
$c_{JJ}^{(es)}$	$C22(\gamma_J,\gamma_K)$	0.0	$C22(\gamma_J,\gamma_K)$
$c_{KK}^{(es)}$	$C22(\gamma_K,\gamma_J)$	$C22(\gamma_K,\gamma_J)$	0.0
$c_{JK}^{(es)}$	C23	0.0	0.0
$c_J^{(es)}$	0.0	0.0	$C23\theta_K$
$c_K^{(es)}$	0.0	$C23\theta_J$	0.0
$c_0^{(es)}$	0.0	$C22(\gamma_J,\gamma_K)\theta_J^2$	$C22(\gamma_K,\gamma_J)\theta_K^2$
$Q_J^{(es)}$	$Q2(\gamma_J,\gamma_K)$	0.0	$Q2(\gamma_J,\gamma_K)$
$Q_K^{(es)}$	$Q2(\gamma_K,\gamma_J)$	$C22(\gamma_K,\gamma_J)$	0.0